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Optimal bounds for the spectral variation of two regular matrix pairs

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Abstract

For the generalized eigenvalue problem, we establish upper bounds for the spectral variation of two regular matrix pairs some of which are optimal. We describe the set of regular matrix pairs for which the bounds are attained.

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1. Introduction and notation

In this note, we shall consider upper bounds for $S_Z(W)$, the spectral variation of Z with respect to W (see (1.5) for definition). For this problem, there have been several theorems, e.g., [2–7]. Here we derive new upper bounds for $S_Z(W)$, only in terms of the spectral norm and determinant of Z and W. Thus our estimates are much less expensive to compute. They are general and available for any regular matrix pairs (see (1.4) for definition).

Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices and $\mathbb{C} = \mathbb{C}^1$. We use capital letters for matrices, lowercase letters for column vectors and $\| \ \|$ both the Euclidean vector norm and the spectral norm. A^H stands for the conjugate transpose of A. I is the unit matrix. For $A, C \in \mathbb{C}^{n\times n}$,

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let A and C have the eigenvalues $\{\lambda_i\}$ and $\{\mu_i\}$ respectively. $S_A(C) = \max_i \min_j |\lambda_j - \mu_i|$ is the spectral variation of C with respect to A.

Theorem 1.1 (Elsner [1]). *For $A, C \in \mathbb{C}^{n\times n}$*

$$S_A(C) \leqslant (\|A\| + \|C\|)^{1-\frac{1}{n}} \|A - C\|^{\frac{1}{n}}. \tag{1.1}$$

Theorem 1.2 (Elsner [1]). *For $A, C \in \mathbb{C}^{n\times n}$, $A \neq 0, C \neq 0$, the following are equivalent*:

(i) $S_A(C) = (\|A\| + \|C\|)^{1-\frac{1}{n}} \|A - C\|^{\frac{1}{n}}$,
(ii) $\exists \varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$ *such that* $A = \varepsilon\|A\|I$, *and* B *has eigenvalue* $-\varepsilon\|B\|$.

In [2], Stewart has showed that a similar result holds for the generalized eigenvalue problem. In order to cite the result by Stewart, we need the number

$$\gamma(A, B) = \max_{|\alpha|^2+|\beta|^2=1} \sigma_{\min}(\beta A - \alpha B), \tag{1.2}$$

where $\sigma_{\min}(X)$ denotes the smallest singular value of X.

Theorem 1.3 (Stewart [2]). *Let $Z = (A, B)$ and $W = (C, D)$ be regular matrix pairs. Then*

$$S_Z(W) \leqslant \frac{1}{\gamma(A, B)} (\|A\|^2 + \|B\|^2)^{\frac{1}{2}(1-\frac{1}{n})} (\|A - C\|^2 + \|B - D\|^2)^{\frac{1}{2n}}. \tag{1.3}$$

The purpose of this note is to obtain a theorem for the generalized eigenvalue problem, corresponding to Elsner's theorem.

First we give the upper bounds for $S_Z(W)$. Then, we describe the set of matrix pairs Z and W for which some of the upper bounds are attained. Finally we show that Theorems 1.1 and 1.3 can be derived by our result.

Let $A, B \in \mathbb{C}^{n\times n}$. We call $Z = (A, B)$ a regular matrix pair if

$$\det(A - \lambda B) \not\equiv 0, \quad \lambda \in \mathbb{C}. \tag{1.4}$$

If $(\alpha, \beta) \neq 0$ and $\det(\beta A - \alpha B) = 0$, we call (α, β) an eigenvalue of Z.

Hereafter, we use $Z = (A, B)$ and $W = (C, D)$ for two regular matrix pairs with eigenvalues (α_i, β_i) and (γ_i, δ_i) respectively. We define the spectral variation of W with respect to Z by

$$S_Z(W) = \max_i \min_j \frac{|\alpha_j\delta_i - \beta_j\gamma_i|}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}\sqrt{|\gamma_i|^2 + |\delta_i|^2}}. \tag{1.5}$$

For convenience, we assume that

$$S_Z(W) = \min_j \frac{|\alpha_j\delta_1 - \beta_j\gamma_1|}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}\sqrt{|\gamma_1|^2 + |\delta_1|^2}} = \min_j \frac{|\alpha_j\delta - \beta_j\gamma|}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}} \tag{1.6}$$

where $\gamma = \frac{\gamma_1}{\sqrt{|\gamma_1|^2+|\delta_1|^2}}$, $\delta = \frac{\delta_1}{\sqrt{|\gamma_1|^2+|\delta_1|^2}}$.

We will use the following metric for Z and W:

$$d_2(Z, W) = \|Z^H(ZZ^H)^{-1}Z - W^H(WW^H)^{-1}W\|.$$

For $d_2(Z, W)$, Elsner and Sun [5] have showed that

$$d_2(Z, W) = \min\left\{\|(ZZ^H)^{-\frac{1}{2}}Z - TW\|, T \in \mathbb{C}^{n\times n}\right\}. \tag{1.7}$$

Let

$$D(Z) \equiv \Big(\max_{|\alpha|^2+|\beta|^2=1} |\det(\beta A - \alpha B)| \Big)^{\frac{1}{n}}. \tag{1.8}$$

2. Results

Lemma 2.1 (Stewart [3]). *There exist unitary matrices $U_1, U_2 \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} U_1^H A U_2 &= \begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}, \\ U_1^H B U_2 &= \begin{pmatrix} \beta_1 & & * \\ & \ddots & \\ 0 & & \beta_n \end{pmatrix}, \end{aligned} \tag{2.1}$$

where (α_i, β_i) are eigenvalues of $Z = (A, B)$ which may be made to appear in any order.

By Lemma 2.1, we have

Lemma 2.2

(i) $D(Z)^n \leqslant \big(\prod_{i=1}^n (|\alpha_i|^2 + |\beta_i|^2)\big)^{\frac{1}{2}}$,

(ii) $D(Z)^n = \big(\prod_{i=1}^n (|\alpha_i|^2 + |\beta_i|^2)\big)^{\frac{1}{2}}$ *if and only if the (α_i, β_i) are linearly dependent on each other.*

Proof. (i) By $|\alpha|^2 + |\beta|^2 = 1$, and Cauchy's inequality, we have

$$|\det(\beta A - \alpha B)| = \prod_{i=1}^n |(\beta\alpha_i - \alpha\beta_i)| \leqslant \left(\prod_{i=1}^n (|\alpha_i|^2 + |\beta_i|^2) \right)^{\frac{1}{2}}. \tag{2.2}$$

That is $D(Z)^n \leqslant \big(\prod_{i=1}^n (|\alpha_i|^2 + |\beta_i|^2)\big)^{\frac{1}{2}}$.

(ii) Equality in Cauchy's inequality (2.2) holds if and only if (α_i, β_i) and $(\bar{\beta}, -\bar{\alpha})$ are linearly dependent on each other, i.e., the (α_i, β_i) are linearly dependent on each other. □

Along the lines of Elsner's proof, we have

Theorem 2.3

$$S_Z(W) \leqslant \frac{1}{D(Z)} \parallel Z \parallel^{1-\frac{1}{n}} \parallel Z - W \parallel^{\frac{1}{n}}. \tag{2.3}$$

Proof. For $W = (C, D)$ by Lemma 2.1, there exist unitary matrices $V_1, V_2 \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} V_2^H C V_1 &= \begin{pmatrix} \gamma_1 & & * \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix}, \\ V_2^H D V_1 &= \begin{pmatrix} \delta_1 & & * \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}. \end{aligned} \tag{2.4}$$

Write $V_1 = (v_1, \ldots, v_n)$. Combining this with (1.6) and (2.4) gives us

$$\delta C v_1 = \gamma D v_1 \tag{2.5}$$

or

$$W \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix} = 0. \tag{2.6}$$

Then

$$\begin{aligned} S_Z(W)^n &= \left(\min_j \frac{|\alpha_j \delta - \beta_j \gamma|}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}} \right)^n \leqslant \prod_{j=1}^n \frac{|\alpha_j \delta - \beta_j \gamma|}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}} \\ &= \frac{|\det(\delta A - \gamma B)(v_1, v_2, \ldots, v_n)|}{\left(\prod_{j=1}^n (|\alpha_j|^2 + |\beta_j|^2) \right)^{\frac{1}{2}}} \qquad (2.7) \\ &\leqslant \frac{1}{D(Z)^n} \left\| Z \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix} \right\| \left\| Z \begin{pmatrix} \delta v_2 \\ -\gamma v_2 \end{pmatrix} \right\| \cdots \left\| Z \begin{pmatrix} \delta v_n \\ -\gamma v_n \end{pmatrix} \right\| \\ &\quad \text{(Hadamard's inequality)} \\ &= \frac{1}{D(Z)^n} \left\| (Z - W) \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix} \right\| \left\| Z \begin{pmatrix} \delta v_2 \\ -\gamma v_2 \end{pmatrix} \right\| \cdots \left\| Z \begin{pmatrix} \delta v_n \\ -\gamma v_n \end{pmatrix} \right\| \\ &\leqslant \frac{1}{D(Z)^n} \|Z - W\| \|Z\|^{n-1}. \qquad \square \qquad (2.8) \end{aligned}$$

Corollary 2.4

$$\text{(i)} \quad S_Z(W) \leqslant \frac{1}{D(Z)} (\|A\|^2 + \|B\|^2)^{\frac{1}{2}(1-\frac{1}{n})} (\|A - C\|^2 + \|B - D\|^2)^{\frac{1}{2n}}, \tag{2.9}$$

$$\text{(ii)} \quad S_Z(W) \leqslant \frac{1}{D(Z)} \|Z\| d_2(Z, W)^{\frac{1}{n}}. \tag{2.10}$$

Proof. By Theorem 2.3, (i) holds.

From (2.6), we have

$$Z \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix} = (ZZ^H)^{\frac{1}{2}} \left((ZZ^H)^{-\frac{1}{2}} Z - TW \right) \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix}, \tag{2.11}$$

where $T \in \mathbb{C}^{n \times n}$, then

$$\left\| Z \begin{pmatrix} \delta v_1 \\ -\gamma v_1 \end{pmatrix} \right\| \leqslant \|Z\| \left\| \left((ZZ^H)^{-\frac{1}{2}} Z - TW \right) \right\|. \tag{2.12}$$

Substituting (2.12) into (2.8) we get (ii). $\square$

Theorem 2.5

$$\begin{aligned} S_Z(W) \leqslant \frac{1}{D(Z)} &\|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}} (\|A\| \|D\| + \|B\| \|C\|)^{1-\frac{1}{n}} \\ &\times (\|D\| \|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}. \end{aligned} \tag{2.13}$$

Proof. Combining (2.5) and (1.6) we see that

$$|\gamma| = \frac{\|Cv_1\|}{\sqrt{\|Cv_1\|^2 + \|Dv_1\|^2}} \leqslant \frac{\|C\|}{\sqrt{v_1^H C^H C v_1 + v_1^H D^H D v_1}} \leqslant \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}} \|C\|, \tag{2.14}$$

and

$$|\delta| \leqslant \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}} \|D\|. \tag{2.15}$$

Hence

$$S_Z(W)^n \leqslant \frac{|\det(\delta A - \gamma B)(v_1, v_2, \ldots, v_n)|}{\left(\prod_{j=1}^n (|\alpha_j|^2 + |\beta_j|^2)\right)^{\frac{1}{2}}} \tag{2.16}$$

$$\leqslant \frac{1}{D(Z)^n} \|((\delta A - \gamma B) - (\delta C - \gamma D))v_1\| \times \|(\delta A - \gamma B)v_2\| \cdots \|(\delta A - \gamma B)v_n\| \quad \text{(Hadamard's inequality)} \tag{2.17}$$

$$\leqslant \frac{1}{D(Z)^n} (|\delta| \|A - C\| + |\gamma| \|B - D\|)(|\delta| \|A\| + |\gamma| \|B\|)^{n-1}$$

$$\leqslant \frac{1}{D(Z)^n} \|(C^H C + D^H D)^{-1}\|^{\frac{n}{2}} (\|A\| \|D\| + \|B\| \|C\|)^{n-1} \times (\|D\| \|A - C\| + \|C\| \|B - D\|). \quad \square \tag{2.18}$$

Let [6,7]

$$\sigma(A, B) = \sup_{\{U_1, U_2\} \in \mathfrak{U}_{(A,B)}} \min_{1 \leqslant i \leqslant n} \sqrt{|\alpha_i|^2 + |\beta_i|^2},$$

where $\mathfrak{U}_{(A,B)} = \{\{U_1, U_2\} | \{U_1, U_2\}$ are all the unitary matrix pairs satisfying the Eqs. (2.1)$\}$.

Noting that

$$\sigma(A, B) \leqslant \left(\prod_{i=1}^n (|\alpha_i|^2 + |\beta_i|^2)\right)^{\frac{1}{2n}}; \quad \gamma(A, B) \leqslant \left(\prod_{i=1}^n (|\alpha_i|^2 + |\beta_i|^2)\right)^{\frac{1}{2n}},$$

we get a counterpart of Theorem 1.2.

Theorem 2.6. *For $A \neq 0$, $B \neq 0$, $C \neq 0$, $D \neq 0$ the following are equivalent*:

$$\text{(i) } S_Z(W) = \frac{1}{D(Z)} \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}} (\|A\| \|D\| + \|B\| \|C\|)^{1-\frac{1}{n}} \times (\|D\| \|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}, \tag{2.19}$$

$$\text{(ii) } S_Z(W) = \frac{1}{\gamma(A, B)} \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}} (\|A\| \|D\| + \|B\| \|C\|)^{1-\frac{1}{n}} \times (\|D\| \|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}, \tag{2.20}$$

$$\text{(iii) } S_Z(W) = \frac{1}{\sigma(A, B)} \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}} (\|A\| \|D\| + \|B\| \|C\|)^{1-\frac{1}{n}} \times (\|D\| \|A - C\| + \|C\| \|B - D\|)^{\frac{1}{n}}, \tag{2.21}$$

(iv) $\exists\varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$ *such that* $A = \varepsilon\|A\|U$, $B = \|B\|U$, $C = -\varepsilon\|C\|V_C$, $D = \|D\|V_D$ *where* U, V_C, V_D *are unitary matrices and* $(-\varepsilon\|C\|, \|D\|)$ *is an eigenvalue of* W.

Proof. We only prove (i) and (iv) are equivalent. It is easy to get (i) from (iv).

Assume that (i) is satisfied. Equality holds in (2.16)–(2.18). Equality in Hadamard's inequality holds only for orthogonal columns. Then there exist orthogonal and normalized vector $u_1, u_2, \ldots, u_n$ such that

$$\begin{aligned}&(\delta(A - C) - \gamma(B - D))v_1 = (|\delta|\|A - C\| + |\gamma|\|B - D\|)u_1,\\ &(\delta A - \gamma B)v_2 = (|\delta|\|A\| + |\gamma|\|B\|)u_2, \ldots, (\delta A - \gamma B)v_n = (|\delta|\|A\| + |\gamma|\|B\|)u_n,\end{aligned}$$

i.e.

$$\begin{aligned}&u_1^H((\delta A - \gamma B) - (\delta C - \gamma D))v_1 = |\delta|\|A - C\| + |\gamma|\|B - D\|,\\ &u_2^H(\delta A - \gamma B)v_2 = |\delta|\|A\| + |\gamma|\|B\|, \ldots, u_n^H(\delta A - \gamma B)v_n = |\delta|\|A\| + |\gamma|\|B\|;\end{aligned} \tag{2.22}$$

and

$$\delta u_i^H A v_i = |\delta|\|A\|, \quad -\gamma u_i^H B v_i = |\gamma|\|B\|, \quad i = 2, \ldots, n. \tag{2.23}$$

Let $a_{ij} = u_i^H A v_j,\ b_{ij} = u_i^H B v_j\ \ i, j = 1, \ldots, n.$

Since $\gamma \neq 0$, $\delta \neq 0$ (see (2.28)), we have

$$\sum_{j=1}^{n} |a_{ij}|^2 \leqslant \|(a_{ij})\|^2 = \|A\|^2 \leqslant |(a_{ii})|^2, \quad \sum_{j=1}^{n} |a_{ji}|^2 \leqslant |(a_{ii})|^2, \quad i = 2, \ldots, n.$$

Thus $a_{ij} = 0$ $(i \neq j)$ and $b_{ij} = 0$ $(i \neq j)$ as well. (a_{ii}, b_{ii}) are now eigenvalues of Z.

It follows from (2.16)–(2.18) that

$$\delta a_{11} - \gamma b_{11} = |\delta|\|A\| + |\gamma|\|B\|, \tag{2.24}$$

then

$$\delta a_{11} = |\delta|\|A\|, \quad -\gamma b_{11} = |\gamma|\|B\|, \tag{2.25}$$

and

$$a_{ii} = a_{11}, \quad b_{ii} = b_{11}, \quad i = 1, \ldots, n. \tag{2.26}$$

Let $\varepsilon = \frac{a_{11}\overline{b_{11}}}{\|a_{11}\|\|b_{11}\|}$, $U = \frac{b_{11}}{\|b_{11}\|}(u_1, u_2, \ldots, u_n)(v_1, v_2, \ldots, v_n)^H$, then

$$A = \varepsilon\|A\|U, \quad B = \|B\|U$$

and

$$(\gamma, \delta) = \left(-\frac{\overline{b_{11}}}{\|b_{11}\|}|\gamma|, \frac{\overline{a_{11}}}{\|a_{11}\|}|\delta|\right),$$

i.e.

$$(-\varepsilon|\gamma|, |\delta|) = \frac{a_{11}}{\|a_{11}\|}(\gamma, \delta), \tag{2.27}$$

where U is a unitary matrix and $|\varepsilon| = 1$.

On the other hand, (2.14), (2.15) and (2.18) imply

$$|\gamma| = \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}}\|C\|, \quad |\delta| = \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}}\|D\|. \tag{2.28}$$

In view of the fact

$$\frac{1}{\|(C^H C + D^H D)^{-1}\|} = \|C\|^2 + \|D\|^2 \geqslant \|C^H C + D^H D\|,$$

we have

$$C^H C + D^H D = (\|C\|^2 + \|D\|^2)I, \tag{2.29}$$

i.e.

$$C^H C = \|C\|^2 I, \quad D^H D = \|D\|^2 I.$$

Let $V_C = \frac{-\bar{\varepsilon}}{\|C\|} C$, $V_D = \frac{1}{\|D\|} D$, then

$$C = -\varepsilon \|C\| V_C, \quad D = \|D\| V_D,$$

where V_C, V_D are unitary matrices.

From (2.27) and (2.28), we know, $(-\varepsilon\|C\|, \|D\|)$ is an eigenvalue of W. (iv) yields. □

3. Remark

3.1. By means of the limiting procedure used by Stewart [4], we can derive Theorem 1.1.

For $A, C \in \mathbb{C}^{n\times n}$, $t > 0$, let A, C have the eigenvalues $\{\lambda_i\}$ and $\{\mu_i\}$ respectively. The pairs $Z_t = (A, tI)$, $W_t = (C, tI)$ are regular matrix pairs. We have

$$D(Z_t) = \max_{|\alpha|^2+|\beta|^2=1} |\det(\beta A - \alpha t I)| \geqslant t^n.$$

By Theorem 2.5 we have

$$\min_i \frac{|t\mu_j - t\lambda_i|}{\sqrt{|\lambda_i|^2 + t^2}\sqrt{|\mu_j|^2 + t^2}} \leqslant \frac{1}{t} \left\| \left(\frac{1}{t^2} C^H C + I \right)^{-1} \right\|^{\frac{1}{2}} (\|A\| + \|C\|)^{1-\frac{1}{n}} \|A - C\|^{\frac{1}{n}}. \tag{3.1}$$

Multiplying (3.1) by t, we have, by letting $t \to \infty$,

$$S_A(C) \leqslant (\|A\| + \|C\|)^{1-\frac{1}{n}} \|A - C\|^{\frac{1}{n}},$$

i.e. Theorem 1.1.

3.2. Since $\gamma(A, B) \leqslant D(Z)$, from (2.3), (2.9), (2.10) and (2.13), we have

$$\text{(i) } S_Z(W) \leqslant \frac{1}{\gamma(A, B)} \parallel Z \parallel^{1-\frac{1}{n}} \parallel Z - W \parallel^{\frac{1}{n}}, \tag{3.2a}$$

$$\text{(ii) } S_Z(W) \leqslant \frac{1}{\gamma(A, B)} (\|A\|^2 + \|B\|^2)^{\frac{1}{2}(1-\frac{1}{n})} (\|A - C\|^2 + \|B - D\|^2)^{\frac{1}{2n}}, \tag{3.2b}$$

$$\text{(iii) } S_Z(W) \leqslant \frac{1}{\gamma(A, B)} \|Z\| d_2(Z, W)^{\frac{1}{n}}, \tag{3.2c}$$

$$\text{(iv) } S_Z(W) \leqslant \frac{1}{\gamma(A, B)} \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}} (\|A\|\|D\| + \|B\|\|C\|)^{1-\frac{1}{n}} \times (\|D\|\|A - C\| + \|C\|\|B - D\|)^{\frac{1}{n}}. \tag{3.2d}$$

We can see that (ii) is Theorem 1.3.

Let $\gamma(A, B) = D(Z)$. Then $\exists \alpha_0, \beta_0 \in \mathbb{C}$, $|\alpha_0|^2 + |\beta_0|^2 = 1$, such that $(\beta_0 A - \alpha_0 B) = \gamma(A, B)V = D(Z)V$, where V is a unitary matrix. In fact we know that both singular value and determinant of a matrix are continuously dependent on the elements of the matrix itself (see [8,9]). Let $\gamma(A, B) = \sigma_{\min}(\beta_0 A - \alpha_0 B)$ and $D(Z) = |\det(\tilde{\beta} A - \tilde{\alpha} B)|^{\frac{1}{n}}$, where $|\alpha_0|^2 + |\beta_0|^2 = |\tilde{\alpha}|^2 + |\tilde{\beta}|^2 = 1$. Since $\sigma_{\min}(tA - hB) \leqslant |\det(tA - hB)|^{\frac{1}{n}}$, for any $t, h \in \mathbb{C}$. So $\sigma_{\min}(\beta_0 A - \alpha_0 B) \leqslant |\det(\beta_0 A - \alpha_0 B)|^{\frac{1}{n}} \leqslant |\det(\tilde{\beta} A - \tilde{\alpha} B)|^{\frac{1}{n}} = \sigma_{\min}(\beta_0 A - \alpha_0 B)$, thus $\sigma_{\min}(\beta_0 A - \alpha_0 B) = |\det(\beta_0 A - \alpha_0 B)|^{\frac{1}{n}}$, i.e., $\sigma_{\min}(\beta_0 A - \alpha_0 B) = \|(\beta_0 A - \alpha_0 B)\|$. We have $(\beta_0 A - \alpha_0 B) = \gamma(A, B)V = D(Z)V$, where V is a unitary matrix.

This implies that, usually, $\gamma(A, B) < D(Z)$. Hence apart from the case $\gamma(A, B) = D(Z)$, the new upper bounds are always better.

3.3. It follows from (2.3), (2.9), (2.10) and (2.13) that

$$\text{(i) } S_Z(W) \leqslant \frac{1}{\sigma(A, B)} \| Z \|^{1-\frac{1}{n}} \| Z - W \|^{\frac{1}{n}}, \tag{3.3a}$$

$$\text{(ii) } S_Z(W) \leqslant \frac{1}{\sigma(A, B)} (\|A\|^2 + \|B\|^2)^{\frac{1}{2}(1-\frac{1}{n})} (\|A - C\|^2 + \|B - D\|^2)^{\frac{1}{2n}}, \tag{3.3b}$$

$$\text{(iii) } S_Z(W) \leqslant \frac{1}{\sigma(A, B)} \|Z\| d_2(Z, W)^{\frac{1}{n}}, \tag{3.3c}$$

$$\begin{aligned} \text{(iv) } S_Z(W) \leqslant & \frac{1}{\sigma(A, B)} \|(C^H C + D^H D)^{-1}\|^{\frac{1}{2}} (\|A\|\|D\| + \|B\|\|C\|)^{1-\frac{1}{n}} \\ & \times (\|D\|\|A - C\| + \|C\|\|B - D\|)^{\frac{1}{n}}. \end{aligned} \tag{3.3d}$$

Inequality (i) has been given by Li (see (4.20) of [6]); (iii) improves the following inequalities given by Li (see (4.15) and (4.16) of [6]):

$$S_Z(W) \leqslant (2^n - 1)^{\frac{1}{n}} \frac{\|Z\|}{\sigma(A, B)} d_2(Z, W)^{\frac{1}{n}}, \tag{3.4}$$

$$S_Z(W) \leqslant n^{\frac{1}{n}} \frac{\|Z\|_{\mathrm{F}}}{\sigma(A, B)} \|Z^H (Z Z^H)^{-1} Z - W^H (W W^H)^{-1} W\|_{\mathrm{F}}^{\frac{1}{n}}, \tag{3.5}$$

where $\| \|_{\mathrm{F}}$ denotes Frobenius norm.

We note that both $\sigma(A, B)$ and $\gamma(A, B)$ are not easy to calculate; upper bounds for $S_Z(W)$ in [3–5] are available only for the matrix pairs on which there are some restrictions.

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